The Relationships Between Qualitative Notions of Optimality for Decision Making Under Logical Uncertainty

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Abstract. We consider decision making problems of the following form: we have a finite set A of decisions and a S of scenarios, along with a total pre-order on decisions in each scenario. With one kind of interpretation it is assumed that one of the scenarios is (or will be) the correct one, but it is not known which, and the ordering in each scenario represents the decision maker's preferences over decisions given that that scenario is the true one. Other interpretations of this setup are also possible, with, for example, the scenarios corresponding to criteria in a multi-criteria decision making problem. We analyse a dozen related notions of optimal decision in this situation, and precisely characterise the subclass relationships between them. Here we consider only purely qualitative notions—in particular, we do not convert the total pre-orders to numerical functions on A—and we assume no uncertainty information, regarding the relative likelihoods of different scenarios. We also show how the subclass hierarchy simplifies under three different conditions: (i) if there exists a decision that is optimal in all scenarios; (ii) if for any decision there exists a most favourable scenario; and (iii) if the orders are all total orders.

1 Introduction

When supporting a decision maker to solve a problem, it can be useful to be able to select a subset of the possible decisions, which are regarded as the optimal ones. When the problem involves uncertainty, there is more than one possible scenario to consider, and when the decision problem is under complete uncertainty, there is no information as to which scenarios are more likely than others. In this paper we look at some possible notions of optimality that may be used to choose the set of optimal decisions when making decisions under complete uncertainty, and discuss the relationships between these notions and between the resulting (possibly empty) optimal sets.

More specifically, we consider decision making problems of the following form: we have a finite set A of decisions and a set S of scenarios, along with a total pre-order $\succcurlyeq_s$ on decisions in each scenario $s \in S$. The main interpretation we will be focusing on is where the scenarios represent the different possible states, so that exactly one of the scenarios represents the true situation (or the state that turns out to be the case). However, the results also all apply under different interpretations; in particular, in a multi-criteria decision making, with a scenario then representing a criterion; or in a group decision making context, where scenarios correspond to agents, with their orderings over decisions.

In this paper, we only consider purely qualitative notions of optimality. In particular, we do not assume that all scenarios have equal probability (or weight or importance). One can always map a total pre-order, in a given scenario s, to a numerical ranking in some canonical way, with e.g., the best element being assigned the number 1, the next best, 2, and so on. However, we do not use such a mapping, because there are an infinite number of other ways of assigning the numbers to decisions in that scenario that respect the total pre-order $\succcurlyeq_s$. Thus we assume weaker information than that assumed in the most similar work such as [2],[8],[10], and[11].

The results in the paper still apply if one has stronger information than that described above, such as a utility function in each scenario, or some uncertain information regarding the relative likelihood of different scenarios (or importance, if the scenarios represent different criteria).

Section 2 discusses some related work. Section 3 defines the different notions of optimality that we consider in this paper. Section 4 gives the main result in the paper, which precisely describes the subclass relationships between the different notions of optimality. Section 5 shows how these relationships simplify under three separate assumptions: (i) when there exists a necessarily optimal element; (ii) when there exists a best scenario for each decision; and (iii) when each scenario totally orders the set of decisions. Section 6 discusses other considerations related to these notions of optimality.

2 Related Work

The work in this paper was partly inspired by the work in [10] concerning notions of optimality in the context of interval-valued soft constraints. In particular, the classes NO and PO (see Section 3.2), appear in [10] (in their more specific context), as does CD, under the name NSDTOP; also, when equivalence $\equiv$ is just equality, CSD and NSO are the same as NDTOP and PDTOP, respectively.

There are a number of different literatures across a wide range of fields and sciences that look at decision making under complete uncertainty or ignorance. However, other work in this area either assumes stronger and less qualitative information, such as some form of utility function over decisions in each scenario, or converts the qualitative information into quantitative information. In particular, we do not assume any information about comparisons between, on the one hand, how good a decision α is in a scenario s, and, on the other hand, how good a decision β is in a different scenario s'. The other work we are aware of in this area assumes such comparison information (explicitly or implicitly).

Some classic decision theory literature includes [2], (building on earlier works [7] and [16]), which developed a set of properties for rational decision making under ignorance. Other works that developed axiomisations and characterisations of individual decision making under ignorance include [3] [8] and [11], while [15] looked at the application to collective decision making and social choice. More recently, [14] distinguished between strict uncertainty, where a set of scenarios is known, and complete ignorance, where there is no information at all on the available states, and both [9] and [14] looked at characterisations of some optimality criteria for complete ignorance.

Some works that looked at various decision rules and optimality notions for decision making under complete uncertainty include [1] and [6], which looked at a characterisation of four different decision rules, namely max-min, min-max, and lexicographical variants of both. [4] looked at maximin, leximin, and a leximin variant called the protective criterion (also characterised by [9]), while [5] and [13] developed some median based approaches.

3 Qualitative Notions of Optimality

Basic Definitions of Orderings: Let $\succcurlyeq$ be a relation on a set A. $\succcurlyeq$ is a *pre-order* if it is reflexive and transitive, and is a *total pre-order* if it is also complete, i.e., if for all $\alpha, \beta \in A$, either $\alpha \succcurlyeq \beta$ or $\beta \succcurlyeq \alpha$ (or both). A total order $\succeq$ is a total pre-order that is antisymmetric, i.e., such that if $\alpha \succeq \beta$ and $\beta \succeq \alpha$ then $\alpha = \beta$. For pre-order $\succcurlyeq$, we define the corresponding strict relation $\succ$ by $\alpha \succ \beta$ if and only if $\alpha \succcurlyeq \beta$ and $\beta \not\succcurlyeq \alpha$. We also define the corresponding equivalence relation $\equiv$ by $\alpha \equiv \beta$ if and only if $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \alpha$.

3.1 Multiple-Ordering Decision Structures

We define a *multiple-ordering decision structure (MODS)* $\mathcal{G}$ to be a tuple $\langle A, S, \{\succcurlyeq_s : s \in S\}\rangle$, where A is a non-empty finite set, known as the set of decisions, S (the set of scenarios) is a non-empty set (that can be finite or infinite), and, for each $s \in S$, relation $\succcurlyeq_s$ is a total pre-order on A. The strict part of $\succcurlyeq_s$ is written as $\succ_s$, and the corresponding equivalence relation is written $\equiv_s$.

Each element of A is interpreted as a decision (or option/choice/alternative) that is available to the decision maker. We will mainly consider that $\mathcal{G}$ relates to a decision making situation under uncertainty, where there is rather weak, qualitative, information. We have a set of options A, and we know how these options are ordered in each of a number of possible scenarios. We assume that exactly one of these scenarios is the correct one, but we have no uncertainty information indicating how likely any of the scenarios are to be the correct scenario. We might say that we have complete ignorance about S, although this could be somewhat misleading, since we may have initially had a much larger set of scenarios S', and have received logical information that enabled us to narrow S' down to S.

Some basic notions associated with MODS $\mathcal{G}$: We make the following definitions, relative to some MODS $\mathcal{G} = \langle A, S, \{\succcurlyeq_s : s \in S\}\rangle$, and where α and β are some arbitrary elements of A.

- We say that α *necessarily dominates* β, written $\alpha \succcurlyeq_N \beta$, if for all $s \in S$, $\alpha \succcurlyeq_s \beta$. Thus $\succcurlyeq_N$ is the intersection of $\succcurlyeq_s$ over all $s \in S$.
- We define $\succ_N$ to be the strict part of $\succcurlyeq_N$.
- We define $\equiv$ to be the equivalence relation associated with $\succcurlyeq_N$. If $\alpha \equiv \beta$ then we say that α and β are *necessarily equivalent* (sometimes abbreviated to *equivalent*). $\equiv$ is the intersection of $\equiv_s$ over all $s \in S$.
- We say that α *necessarily strictly dominates* β, written $\alpha \succ_{NS} \beta$, if and only if for all $s \in S$, $\alpha \succ_s \beta$. Thus $\succ_{NS}$ is the intersection of $\succ_s$ over all $s \in S$.

Let s be a scenario in S. We define O_s and SO_s as follows.

- O_s is the set of optimal elements in s, i.e., the set of $\alpha \in A$ such that for all $\beta \in A$, $\alpha \succcurlyeq_s \beta$.
- SO_s is the set of strictly optimal elements in s, i.e., the set of $\alpha \in A$ such that $\alpha \succ_s \beta$ for all $\beta \in A$ with $\beta \not\equiv \alpha$.

3.2 The basic optimality classes

We define six basic classes, representing different notions of optimality. Further optimality classes are defined in Sections 3.5 and 3.6. We say that α is *necessarily optimal* if α necessarily dominates every $\beta \in A$, i.e., if $\alpha \succcurlyeq_N \beta$ for all $\beta \in A$. Thus α is necessarily optimal if and only if it is optimal in every scenario, corresponding to the case of unanimity. α is *necessarily strictly optimal* if α necessarily strictly dominates every β that is not equivalent to α, i.e., if $\alpha \succ_{NS} \beta$ for all $\beta \in A$ such that $\beta \not\equiv \alpha$.

$\mathrm{NO}(\mathcal{G})$ is the set of necessarily optimal elements.

$\mathrm{NSO}(\mathcal{G})$ is the set of necessarily strictly optimal elements.

$\mathrm{PO}(\mathcal{G})$ is the set of possibly optimal elements, i.e., elements that are optimal in some scenario. Thus $\mathrm{PO}(\mathcal{G})$ is the set of $\alpha \in A$ such that $\exists s \in S$, $\forall \beta \in A$: $\alpha \succcurlyeq_s \beta$.

$\mathrm{PSO}(\mathcal{G})$ is the set of possibly strictly optimal elements, i.e., the set of $\alpha \in A$ such that α is strictly optimal in some scenario. Hence $\alpha \in \mathrm{PSO}(\mathcal{G})$ if and only if there exists $s \in S$ such that $\alpha \succ_s \beta$ for all $\beta \in A$ with $\beta \not\equiv \alpha$.

$\mathrm{CD}(\mathcal{G})$: $\alpha \in \mathrm{CD}(\mathcal{G})$ if and only if α *can dominate* any other decision, i.e., if and only if $\forall \beta \in A$, $\exists s \in S : \alpha \succcurlyeq_s \beta$. It can be seen that $\alpha \notin \mathrm{CD}(\mathcal{G})$ if and only if $\exists \beta \in A$ such that $\beta \succ_{NS} \alpha$, so $\mathrm{CD}(\mathcal{G})$ are the decisions that are undominated with respect to $\succ_{NS}$.

$\mathrm{CSD}(\mathcal{G})$: $\alpha \in \mathrm{CSD}(\mathcal{G})$ if and only if it *can strictly dominate* any non-equivalent decision, i.e., for all $\beta \in A$ such that $\beta \not\equiv \alpha$, there exists $s \in S$ such that $\alpha \succ_s \beta$. $\mathrm{CSD}(\mathcal{G})$ are the decisions that are undominated with respect to $\succ_N$.

When the intended choice of $\mathcal{G}$ is clear, we'll abbreviate $\mathrm{NO}(\mathcal{G})$ to just NO, and similarly, for the other classes. We also define $\mathrm{NOPSO} = \mathrm{NO} \cap \mathrm{PSO}$ and $\mathrm{PO}' = \mathrm{PO} \cap \mathrm{CSD}$.

Classes NO, PO, NSO and PSO can be expressed in terms of the optimal and strictly optimal sets O_s and SO_s: $\mathrm{NO} = \bigcap_{s \in S} \mathrm{O}_s$; $\mathrm{PO} = \bigcup_{s \in S} \mathrm{O}_s$; $\mathrm{NSO} = \bigcap_{s \in S} \mathrm{SO}_s$; $\mathrm{PSO} = \bigcup_{s \in S} \mathrm{SO}_s$.

Example 1. Consider MODS $\mathcal{G}$ with set of decisions $A = \{\alpha, \beta, \gamma, \delta\}$, set of scenarios $\{s_1, s_2, s_3\}$, where the associated total pre-orders are given by Table 1. For example, the scenario s_1 has associated total pre-order $\succcurlyeq_{s_1}$ over $A = \{\alpha, \beta, \gamma, \delta\}$, given by the transitive closure of $\alpha \equiv_{s_1} \beta \succcurlyeq_{s_1} \gamma \succcurlyeq_{s_1} \delta$. Thus O_{s_1}, the set of optimal elements in s_1, is equal to $\{\alpha, \beta\}$. α and β are optimal in the first two scenarios, and δ is optimal in the third. Hence $\mathrm{PO} = \{\alpha, \beta, \delta\}$, since these are the decisions that are optimal in some scenario. Because no decision is optimal in all the scenarios, there is no necessarily optimal decision, i.e., $\mathrm{NO} = \emptyset$, and so $\mathrm{NSO} = \emptyset$ also. No two decisions are equivalent (in all scenarios), so $\equiv$ is just equality. δ is strictly optimal in the third scenario, and $\mathrm{PSO} = \{\delta\}$. $\mathrm{CSD} = \{\beta, \gamma, \delta\}$; $\alpha \notin \mathrm{CSD}$ since in none of the three scenarios does it strictly dominate β. Therefore, $\mathrm{PO}' = \mathrm{PO} \cap \mathrm{CSD} = \{\beta, \delta\}$. Finally, $\mathrm{CD} = \{\alpha, \beta, \gamma, \delta\}$ since no decision is strictly dominated by another in all scenarios.

3.3 Relations between basic classes

We say that X is *closed under improvement* if the following property holds, for all $\alpha, \beta \in A$: If $\alpha \in \mathrm{X}$ and $\beta \succcurlyeq_N \alpha$ then $\beta \in \mathrm{X}$. NSO, NOPSO, PO', PO, CSD and CD, as well as all the other classes we consider in this paper, are closed under improvement,

Proposition 1. *Let* X *be a subset of* A *that is closed under improvement.*

(i) $\mathrm{NO} \cap \mathrm{X} \neq \emptyset$ *implies* $\mathrm{NO} \subseteq \mathrm{X}$*; hence, if also* $\mathrm{X} \subseteq \mathrm{NO}$ *then* $\mathrm{X} = \mathrm{NO}$.
(ii) $\mathrm{X} \neq \emptyset$ *implies* $\mathrm{X} \cap \mathrm{CSD} \neq \emptyset$.

The following result gives the basic subset relationships between the classes we've introduced, where the notation $A \subseteq (B, C) \subseteq D$ means $A \subseteq B \subseteq D$ and $A \subseteq C \subseteq D$.

Proposition 2. PO'*,* PO*,* CSD *and* CD *are always non-empty. The classes satisfy the following relationships:*

$$\mathrm{NSO} \subseteq \mathrm{NOPSO} \subseteq (\mathrm{NO}, \mathrm{PSO}) \subseteq \mathrm{PO}' \subseteq (\mathrm{PO}, \mathrm{CSD}) \subseteq \mathrm{CD}.$$

Table 1. Some scenarios and their associated orderings.

$\succcurlyeq_{s_1}$	$\succcurlyeq_{s_2}$	$\succcurlyeq_{s_3}$	$\succcurlyeq_{s_4}$	$\succcurlyeq_{s_5}$	$\succcurlyeq_{s_6}$	$\succcurlyeq_{s_7}$
$\alpha\ \beta$	$\alpha\ \beta$	δ	$\alpha\ \gamma$	$\alpha\ \beta\ \delta$	γ	γ
γ	δ	γ	β	γ	β	δ
δ	γ	β	δ		α	α
		α			δ	β

3.4 Discussion of the basic classes

NO consists of all decisions that are optimal in every scenario. Of course, NO will often be empty. If it is non-empty then the subclass graph collapses into a chain (see Proposition 5 below). NSO will almost always be empty. In fact it's empty unless exactly the same decisions are optimal in every scenario, so that all scenarios agree on which decisions are optimal. In that case all the classes collapse into one class. PSO is non-empty if and only if in some scenario the optimal elements are all necessarily equivalent.

PO contains the decisions that are optimal in some scenario. For decision making under uncertainty, where some unknown scenario ordering s gives the (in some sense) correct ordering on decisions, the possibly optimal elements are the ones that could be the best.

If α is not in CD then there exists some decision that is strictly better in every scenario. Thus, being a member of CD is a very weak notion of optimality, and it would hard to argue that any element that is not in CD should be viewed as being an optimal element. One might also argue that being a member of CSD is a minimal requirement for it to be considered as any kind of optimal element. The reason is that if $\alpha \notin \mathrm{CSD}$ then there exists some decision β that is at least as good in every scenario, and better in some scenario. If X is non-empty and closed under improvement then $\mathrm{X}' = \mathrm{X} \cap \mathrm{CSD}$ is non-empty (Proposition 1(ii)). Hence we can reduce any such non-empty class X to the potentially smaller non-empty set X', by eliminating elements not in CSD.

3.5 Maximally Possibly Optimal Decisions

One can refine the notion of *possibly optimal*. Recall that O_s is the set of optimal elements in s, i.e., the set of $\alpha \in A$ such that $\alpha \succcurlyeq_s \beta$ for all $\beta \in A$. For each $\alpha \in A$, define $\mathrm{Opt}(\alpha)$ to consist of the set of scenarios that α is optimal in, i.e., $\{s \in S : \mathrm{O}_s \ni \alpha\}$. Let $\mathrm{Opt}(A) = \{\mathrm{Opt}(\alpha) : \alpha \in A\}$. We say that $\alpha \in \mathrm{MPO}(\mathcal{G})$ (α is *maximally possibly optimal* for $\mathcal{G}$) if α is optimal in a maximal set of scenarios, i.e., if $\mathrm{Opt}(\alpha)$ is a maximal subset (w.r.t. $\subseteq$) in $\mathrm{Opt}(A)$. The definitions immediately imply the following result (abbreviating $\mathrm{MPO}(\mathcal{G})$ to MPO etc.), where $\mathrm{MPO}' = \mathrm{MPO} \cap \mathrm{CSD}$.

Proposition 3. *For all* $\mathcal{G}$, MPO *is non-empty. Also,* $\mathrm{NO}, \mathrm{PSO} \subseteq \mathrm{MPO} \subseteq \mathrm{PO}$, *and* $\mathrm{NO}, \mathrm{PSO} \subseteq \mathrm{MPO}' \subseteq \mathrm{PO}'$.

3.6 Extreme Elements

We define another refinement of *possibly optimal*, based on the notion of extreme solution in multi-criteria optimisation [12]. We can consider the optimal decisions with respect to some scenario s_1, and then consider maximal elements of this set with respect to another scenario s_2; this can be continued until the set of elements are all equivalent.

Let $s_1, \ldots, s_k$ be a finite sequence of scenarios. Define $A_{s_1} = \max_{\succcurlyeq_{s_1}} A = \mathrm{O}_{s_1}$. For $i = 2, \ldots, k$, we then inductively define $A_{s_1,\ldots,s_i}$ to be $\max_{\succcurlyeq_{s_i}} A_{s_1,\ldots,s_{i-1}}$.

Define the set EXT of extreme elements as follows. $\alpha \in \text{EXT}$ if and only if there exists a finite sequence $s_1, \ldots, s_k$ of scenarios such that all elements in $A_{s_1,\ldots,s_k}$ are equivalent (with respect to $\equiv$), and $\alpha \in A_{s_1,\ldots,s_k}$.

Proposition 4. EXT *is always non-empty.* $\text{NO}, \text{PSO} \subseteq \text{EXT} \subseteq \text{PO}, \text{CSD}$*, so* $\text{EXT} \subseteq \text{PO}'$*. Furthermore,* $\text{MPO-EXT} = \text{MPO} \cap \text{EXT}$ *is always non-empty.*

4 Subclass Relationships

Here we show the precise subclass relationship between the different definitions of optimality. In the result below we define, for any $\mathcal{G}$, $\emptyset(\mathcal{G})$ to be $\emptyset$, and $\text{A}(\mathcal{G})$ to be A, the set of decisions in $\mathcal{G}$.

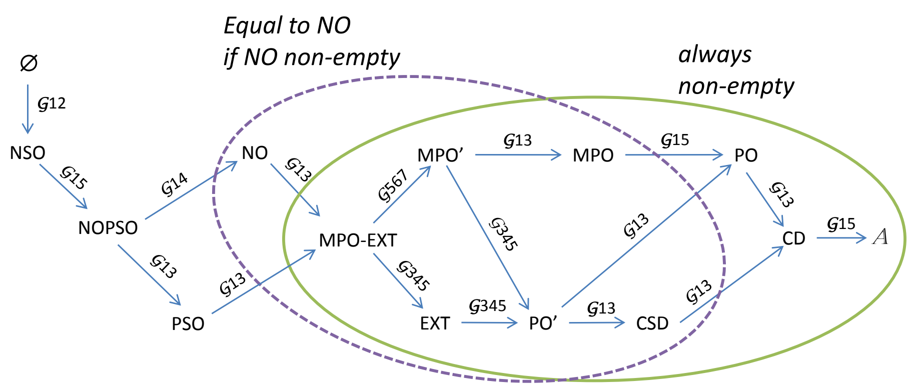

Fig. 1. Subclass relationships, including examples showing inequality between adjacent classes.

Theorem 1. *Consider the directed acyclic graph given in Figure 1, including edges* $\emptyset \to \text{NSO}$*,* $\text{NSO} \to \text{NOPSO}$*, and so on. The vertices of the graph are* $\mathcal{C} = \{\emptyset, \text{NSO}, \text{NOPSO}, \text{NO}, \text{PSO}, \text{MPO-EXT}, \text{EXT}, \text{MPO}', \text{MPO}, \text{PO}', \text{PO}, \text{CSD}, \text{CD}, \text{A}\}$*. Let* $\mathcal{L}$ *be the transitive closure of this graph, so that* $(\text{X}, \text{Y}) \in \mathcal{L}$ *if and only if there exists a directed path from* X *to* Y*. Then for any different* $\text{X}, \text{Y} \in \mathcal{C}$*,*

$$(\text{X}, \text{Y}) \in \mathcal{L} \text{ if and only if for all MODS } \mathcal{G}, \ \text{X}(\mathcal{G}) \subseteq \text{Y}(\mathcal{G}).$$

Furthermore, $\text{MPO-EXT}, \text{EXT}, \text{MPO}', \text{MPO}, \text{PO}', \text{PO}, \text{CSD}$ *and* CD *are non-empty for any* $\mathcal{G}$*, but there exists* $\mathcal{G}$ *such that* $\text{NSO}, \text{NOPSO}, \text{NO}$ *and* PSO *are empty.*

5 Subclass Simplifications Under Extra Conditions

We consider three important extra conditions, and show how the subclass relationships simplify considerably under each. We also illustrate these results with three general forms of instance. As usual, we write the MODS $\mathcal{G}$ as $\langle A, S, \{\succcurlyeq_s : s \in S\}\rangle$, and abbreviate $\text{NO}(\mathcal{G})$ to NO, and so on.

Table 2. Classes in Different Examples. We consider six different MODS $\mathcal{G}$ over set of decisions $A = \{\alpha, \beta, \gamma, \delta\}$, using different sets of scenarios from Table 1. For example, $\mathcal{G}12$ involves set of scenarios $\{s_1, s_2\}$ with orderings as defined in Table 1.

	NSO	NOPSO	NO	PSO	MPO-EXT	EXT	MPO′	MPO	PO′	PO	CSD	CD
$\mathcal{G}12$	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β
$\mathcal{G}13$	$\emptyset$	$\emptyset$	$\emptyset$	δ	β, δ	β, δ	β, δ	α, β, δ	β, δ	α, β, δ	β, γ, δ	A
$\mathcal{G}14$	$\emptyset$	$\emptyset$	α	$\emptyset$	α	α	α	α	α	α, β, γ	α	α, β, γ
$\mathcal{G}15$	$\emptyset$	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β	α, β, δ	α, β	α, β, δ
$\mathcal{G}345$	$\emptyset$	$\emptyset$	$\emptyset$	δ	α, δ	α, γ, δ	α, δ	α, δ	A	A	A	A
$\mathcal{G}567$	$\emptyset$	$\emptyset$	$\emptyset$	γ	β, γ, δ	β, γ, δ	A	A	A	A	A	A

5.1 When there exists a necessarily optimal element

The following result considers the case when there exists a necessarily optimal element. Part (i) of the result below implies that the graph of subset relations in Figure 1 collapses to a chain, with NO equalling CSD, and PO = CD. Part (ii) implies that either NOPSO = NO = PSO or NOPSO = $\emptyset$. Part (iii) implies that when there exists a necessarily strictly optimal element, the set of optimal elements are the same in each scenario and all the classes including NSO, NO, PSO, PO, CSD, CD and any O_s and SO_s collapse into one class, i.e., are all equal (see Figure 1).

Proposition 5. *Assume that* NO $\neq \emptyset$, *and let s be an arbitrary scenario in S. Then,*

(i) NSO $\subseteq$ SO_s $\subseteq$ PSO $\subseteq$ (NO = MPO = EXT = CSD) $\subseteq$ O_s $\subseteq$ (PO = CD).
(ii) For X $\in$ {NSO, NOPSO, PSO, SO_s}, *either* X = $\emptyset$ *or* X = NO.
(iii) If NSO $\neq \emptyset$ *then all the classes* NSO, SO_s, PSO, NO, CSD MPO, EXT, O_s, PO *and* CD *are equal.*

5.2 Most favourable scenario for a decision

Given decision α, we say that scenario s' is a *most favourable scenario for* α if for all $\beta \in A$ and for any scenario $s \in S$, $[\alpha \not\prec_s \beta \Rightarrow \alpha \not\prec_{s'} \beta]$ and $[\alpha \succ_s \beta \Rightarrow \alpha \succ_{s'} \beta]$.

Proposition 6. *Suppose, for every decision $\alpha \in A$, there exists a most favourable scenario for α. Then* CSD = PSO = MPO *and* CD = PO. *Hence we then have the following relationships:*

NSO $\subseteq$ NO $\subseteq$ (PSO = EXT = PO′ = CSD = MPO) $\subseteq$ (PO = CD).

5.3 When scenarios totally order decisions

If the ordering in each scenario is a total order then the class hierarchy simplifies in different way.

Proposition 7. *Let $\mathcal{G} = \langle A, S, \{\succcurlyeq_s : s \in S\}\rangle$ be such that, for every scenario s, $\succcurlyeq_s$ is a total order. Then* $\text{NSO} = \text{NO}$*, and* $\text{PSO} = \text{PO}$*, and* $\text{CSD} = \text{CD}$*, and for every scenario s,* $\text{SO}_s = \text{O}_s$*. Hence we have the following relationships between the classes:*
$(\text{NSO} = \text{NOPSO} = \text{NO}) \subseteq (\text{PSO} = \text{PO} = \text{MPO} = \text{EXT}) \subseteq (\text{CSD} = \text{CD})$.
If there exists a necessarily optimal decision α then all the classes NSO*,* NOPSO*,* NO*,* PSO*,* PO$'$*,* PO*,* CSD CD *are equal to $\{\alpha\}$. If, for every decision $\alpha \in A$, there exists a most favourable scenario for α then we have*
$(\text{NSO} = \text{NOPSO} = \text{NO}) \subseteq (\text{PSO} = \text{PO}' = \text{PO} = \text{MPO} = \text{EXT} = \text{CSD} = \text{CD})$.

6 Discussion

If there exists a necessarily optimal element, then being necessarily optimal seems the most natural notion of optimality; most of the optimality classes we consider collapse to NO in this case: see Proposition 5. However, more generally, there is no unique most appropriate notion of optimality in all contexts. If one of the scenarios represents the true preference ordering, then PO gives the decisions that could be optimal. However, we might eliminate decisions not in CSD (each of which is inferior to one that is in CSD), leading to PO$'$. Where this class is large, we may refine this to consider more special elements in different ways, and based on different intuitions, leading to MPO$'$, EXT and MPO-EXT, the latter being the most specific class we consider that is always non-empty.

Suppose that we wish to show a list of decisions to the decision maker, with the most interesting ones first. An important question is: which decisions should we show first? We may have stronger information than the MODS (and the results in the paper still hold). However, suppose that we just have the qualitative information expressed by the MODS $\mathcal{G} = \langle A, S, \{\succcurlyeq_s : s \in S\}\rangle$, and that we only want to use this purely qualitative information (e.g., we don't want to convert the orderings to numerical rankings, and we don't want to reason based on numbers of the scenarios). One approach is to associate each decision $\alpha \in A$ with the minimal class $\text{X}(\alpha)$ that contains α, among the classes $\mathcal{C}$ in Figure 1. (This is unique because $\mathcal{C}$ is closed under intersection.) We then define relation $\succ_{\mathcal{L}}$ on A by $\alpha \succ_{\mathcal{L}} \beta$ if and only if $\text{X}(\alpha)$ is a strict subset of $\text{X}(\beta)$. This is compatible with relation $\succ_N$, and the union of the two relations, $\succ_{\mathcal{L}} \cup \succ_N$ is acyclic (and, in fact, transitive). We can therefore show the user decisions in an order compatible with $\succ_{\mathcal{L}} \cup \succ_N$.

In this paper we only considered qualitative orderings, and purely qualitative notions of optimality. If in each scenario there is instead a numerical utility function over decisions (or if one converts the qualitative orderings to numerical rankings) then other natural notions of optimality are available, such as minimax regret [17], or maximin. Furthermore, if we have, for example, a probability distribution over scenarios, then we can consider the decisions that have highest probability of being optimal, or those that maximise expected utility.

Interestingly, these considerations are also relevant in the purely qualitative case. For example, when the set of scenarios is finite, we can consider the set of decisions that have highest probability of being optimal under some strictly positive probability distribution. It can be shown that this equals MPO. Similarly, we can consider the set of decisions that minimise maximum regret, under some compatible assignment of utility functions to scenarios.

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References

1. Arlegi Pérez, R.: Rational evaluation of actions under complete uncertainty (2001)
2. Arrow, K.J., Hurwicz, L.: An Optimality Criterion for Decision Making under Ignorance. Basil Blackwell, Oxford (1972)
3. Atkinson, F.V., Church, J.D., Harris, B.: Decision procedures for finite decision problems under complete ignorance. The Annals of Mathematical Statistics vol. 35(4), pp. 1644–1655 (1964)
4. Barbara, S., Jackson, M.: Maximin, leximin, and the protective criterion: Characterizations and comparisons. Journal of Economic Theory vol. 46(1), pp. 34–44 (1988)
5. Bhattacharyya, A.: Median-based rules for decision-making under complete ignorance. Working papers, Sam Houston State University, Department of Economics and International Business
6. Bossert, W., Pattanaik, P.K., Xu, Y.: Choice under complete uncertainty: Axiomatic characterizations of some decision rules. Economic Theory vol. 16(2), pp. 295–312 (2000)
7. Chernoff, H.: Rational selection of decision functions. Econometrica vol. 22(4), pp. 422–443 (1954)
8. Cohen, M., Jaffray, J.Y.: Rational behavior under complete ignorance. Econometrica vol. 48(5), pp. 1281–99 (1980)
9. Congar, R., Maniquet, F.: A trichotomy of attitudes for decision-making under complete ignorance. Mathematical Social Sciences vol. 59(1), pp. 15 – 25 (2010)
10. Gelain, M., Pini, M., Rossi, F., Venable, K., Wilson, N.: Interval-valued soft constraint problems. Annals of Mathematics and Artificial Intelligence vol. 58, pp. 1–38 (2010)
11. Hogarth, R.M., Kunreuther, H.: Decision making under ignorance: Arguing with yourself. Journal of Risk and Uncertainty vol. 10(1), pp. 15–36 (1995)
12. Junker, U.: Preference-based search and multi-criteria optimization. Annals OR vol. 130(1-4), pp. 75–115 (2004)
13. Kannai, Y., Peleg, B.: A note on the extension of an order on a set to the power set. Journal of Economic Theory vol. 32(1), pp. 172 – 175 (1984)
14. Larbi, R.B., Konieczny, S., Marquis, P.: A characterization of optimality criteria for decision making under complete ignorance. In: Lin, F., Sattler, U., Truszczynski, M. (eds.) Principles of Knowledge Representation and Reasoning: Proceedings of the Twelfth International Conference, KR 2010. AAAI Press (2010)
15. Maskin, E.: Decision-making under ignorance with implications for social choice. Theory and Decision vol. 11, pp. 319–337 (1979)
16. Milnor, J.W.: Games against nature. Tech. rep., RAND Corporation (1951)
17. Savage, L.J.: The theory of statistical decision. Journal of the American Statistical Association vol. 46(253), pp. 55–67 (1951)