The Bounded Transitive Closure Problem

Luis Quesada

Given the directed graphs g_{min}, g_{max}, tcg_{min} and tcg_{max}, The Bounded Transitive Closure Problem (BTC) is to find a directed graph g such that:

$$\begin{array}{c} g_{min} \subseteq g \subseteq g_{max} \\ \text{and} \\ tcg_{min} \subseteq TransClos(g) \subseteq tcg_{max} \end{array} \tag{1}$$

Let us show that BTC is NP complete by reducing The Disjoint Path Problem (DP) to BTC. The k-Disjoint-paths problem consist in finding k pairwise disjoint paths between k pairs of nodes $\langle s_1, d_1\rangle$, $\langle s_2, d_2\rangle$, $\ldots$, $\langle s_k, d_k\rangle$. Both the node-disjoint version and the edge-disjoint version are NP-complete even for $k = 2$ [SP78]. So if we express the problem of (node-disjointedly) connecting $\langle a, b\rangle$ and $\langle c, d\rangle$ in g_{max} in terms of BTC, then we prove that BTC is NP complete.

Let $p_{\langle a,b\rangle}$ and $p_{\langle c,d\rangle}$ be the paths connecting $\langle a, b\rangle$ and $\langle c, d\rangle$ respectively. The first thing to notice is that, if $p_{\langle a,b\rangle}$ and $p_{\langle c,d\rangle}$ share a node k, the graph composed of $p_{\langle a,b\rangle}$ and $p_{\langle c,d\rangle}$ would be a graph where a reaches d and c reaches b. Notice that in order to reach d from a we just need to go from a to k using $p_{\langle a,b\rangle}$, and then from k to d using $p_{\langle c,d\rangle}$.

If we want to avoid that $p_{\langle a,b\rangle}$ and $p_{\langle c,d\rangle}$ share nodes, we need to impose that a does not reach d and c does not reach b. Then, the problem of finding two disjoint paths connecting $\langle a, b\rangle$ and $\langle c, d\rangle$ in g_{max} can be reduced to the following BTC [1]:

$$\begin{array}{rcl} g_{min} & = & \emptyset \\ g_{max} & = & \text{the given graph} \\ tcg_{min} & = & \{\langle a, b\rangle, \langle c, d\rangle\} \\ tcg_{max} & = & TransClos(g_{max}) - \{\langle a, d\rangle, \langle c, b\rangle\} \end{array} \tag{2}$$

If g is a solution of the BTC the disjoint paths can be obtained by running DFS rooted at a and c respectively. Notice that any path found by DFS would be correct since all paths from a to b are pairwise disjoint with all paths from b to d.

[1] In the following equation, we will represent a graph as a set of edges.

References

[SP78] Y. Shiloach and Y. Perl. Finding two disjoint paths between two pairs of vertices in a graph. *Journal of the ACM*, 1978.